Int. J. Multiphase Flow Vol. 7, No. 6, pp. 619–633, 1981
Printed in Great Britain.

0301-9322/81/060619-15$02.00/0
Pergamon/Elsevier

VIRTUAL DENSITY AND SPEED OF SOUND IN A FLUID–SOLID MIXTURE WITH PERIODIC STRUCTURE

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(*Received* 26 *October* 1980; *in revised form* 15 *May* 1981)

Abstract—In a two-phase mixture of solid particles in an inviscid compressible fluid the effective density of the mixture differs from the average density due to relative accelerations between the phases. The effective density directly influences the effective speed of sound. A constructive theory is given to compute density tensors which describe the inertial coupling between the two phases. The theory is based on Hamilton's principle and a general homogenization method as described in a previous paper. Here we describe a variant of the theory which clarifies the differences between surface and volume averaged velocities. This difference is important in case of two-phase configurations showing a regular periodic pattern like rod bundles in fluid. Quantitative results are given for rod bundles and partly for spheres in regular square arrangements.

1. INTRODUCTION

1.1 *Description of the problem*

In an accelerated mixture of two phases with different densities relative or slip accelerations appear. The acceleration slip depends upon the virtual density (see Drew *et al.* 1979) which couples the accelerations of the two phases. This slip influences the effective density and the effective speed of sound of the mixture. Thus the virtual density is an important parameter but for general two-phase mixtures it is difficult to predict.

In a recent paper (Schumann 1981) linear homogenized equations of motion have been deduced for a bundle of compressible rods with periodic configuration in an acoustical inviscid fluid at small velocities. The theory provides a constructive method to compute effective density tensors and sound speeds. In the previous and the present paper only inertia and pressure forces are taken into account explicitly while all other forces are assumed to be prescribed. The theory is intended to be used to describe pressure waves in the core of a nuclear pressurized water reactor. For this purpose elasticity and friction forces have to be included and appropriate models are given by Schumann & Benner (1981). For high frequency motion (e.g. for $\omega R^2/\nu \gg 1$, ω = characteristic frequency, R = rod radius, ν = kinematic viscosity of the fluid) these forces are small in comparison to the inertia forces. On the other hand, the characteristic frequency of motion ω is assumed to be so small that the fluid in a cell behaves locally incompressible, i.e. $\omega R/a \ll 1$ (a = speed of sound). This does not exclude long wave density variations. The kinetic energy related to rotational degrees of freedom and to local compression of the structural material is assumed to be small in comparison to the kinetic energy related to slip motion between fluid and structure. Further the present theory is restricted to cases where the convective derivatives $u_i \partial/\partial x_i$ are small in comparison to the local derivative $\partial/\partial t$ and where the fluid-structural interface deformations are small in comparison to R. These assumptions are appropriate for small amplitude pressure wave phenomena.

The given homogenization theory allows for arbitrary rod shapes and patterns as long as the pattern is periodic so that one can divide the domain into a lattice of rectangular cells each having the same geometrical and material properties. The side lengths of such a cell are ϵ^1, ϵ^2, ϵ^3 and $\epsilon := \max(\epsilon^i)$ must be small in comparison to the diameter D of the domain V. Neglecting the boundary ∂V of the domain V, one can define the cells in a somewhat arbitrary manner. For example, figure 1 shows different types of cells for a bundle of cylindrical rods in

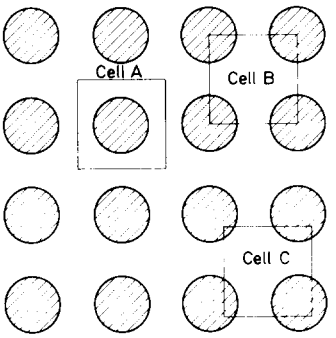

Fig. 1. Three arbitrary choices of selecting a cell in a periodic pattern.

square pattern. The theory was formulated in terms of surface averaged fluid velocities. This has advantages in that the averaged equation of continuity and boundary conditions at ∂V are relatively easy to deduce. The disadvantage, however, is that the resultant homogenized equations are dependent on the cell definition actually chosen.

In this paper, the theory is reformulated in terms of volume averaged fluid velocities. The homogenized equations of motion in terms of volume averaged velocities are independent of the cell definition. The four density tensors introduced in Schumann (1981) are related to one symmetric virtual density tensor. After a coordinate transformation, which diagonalizes the virtual density tensor, the effective speed of sound can be computed explicitly.

The present theory is valid also for structural material which has the form of a porous body or of a cloud of particles floating in a fluid as long as the configuration is periodic. Moreover, the theory is of relevance for general two-phase problems in that it gives the relation between the virtual density and the volume fraction α (also called porosity) of the fluid. But this relevance is qualitative only because the assumed periodicity and small interface deformations are restrictive assumptions in this connection. Nevertheless, the results support and extend the present state of knowledge in this field as will be explained in Section 6.

1.2 *Summary of the homogenization method*

The general homogenization approach follows the proposals of Bensoussan, Lions & Papanicolaou (1978) and Berdichevskii (1977) as follows. The starting point is Hamilton's principle in a form suitable for a heterogeneous mixture. As shown in Schumann (1981) the solutions of the local equations of motion extremize a suitable functional. Then, approximative local solutions are specified in terms of a sum of local trial functions times global amplitudes. The latter are defined such that in the limit $\epsilon/D \to 0$ they represent the (volume or surface) averaged fluid and structural motions. The local trial functions are determined such that they extremize the functional for fixed global amplitudes. Under the assumption of smooth or long-wave global fields it turns out that the local trial functions are solutions of Laplace equations in a single cell with periodic boundary conditions at the cell faces. The solution of Laplace equation is, in principle, a standard numerical task. Thereafter, homogenized equations of motion follow in terms of the global amplitudes from the variational principle for fixed local trial functions and for the limit $\epsilon/D \to 0$. An important restriction is that the global amplitudes have to satisfy the averaged continuity equation. This restriction is taken into account by introducing a Lagrangian multiplier which can be identified as the averaged pressure.

The procedure has been described by Schumann (1981). We do not repeat all the details but state only those changes which are necessary to convert from surface to volume averages. In this paper, equation numbers start at 101. Equation numbers below 100 refer directly to equations in the preceding publication.

2. SURFACE AND VOLUME AVERAGED VELOCITIES

First, some essential definitions are repeated from Schumann (1981). The domain V is composed of rectangular cells $V_{\mathbf{m}}$, see Figure 2. The cells are numbered with the integer vector $\mathbf{m}$. Each cell contains a section $S_{\mathbf{m}}$ of a rod, such that the cells form a periodic lattice, and a section $F_{\mathbf{m}}$ of the fluid.

The notation is such that if y is any quantity then y_i are vector components where repeated lower indices imply summation from one to three. On the other side, y^i is an indexed quantity not implying summation. $y^\epsilon(\mathbf{x}, t)$ is a local field quantity as a function of space and time, $y'(\mathbf{x}, t)$ is a local ansatz function which approximates $y^\epsilon(x, t)$, $\bar{y}(\mathbf{m}, t)$ is an averaged quantity within cell $\mathbf{m}$ and $y(x, t)$, will denote corresponding homogenized fields. A star is used as upper index for volume averaged quantities and $\dot{y} \equiv \partial y / \partial t$.

The ansatz for the local fluid velocity u_i^ϵ and the local structural deflection w_i^ϵ $(i = 1, 2, 3)$ defined in [12, 13] is

$$u_i^\epsilon(\mathbf{x}, t) \approx u_i'(\mathbf{x}, t), \qquad w_i^\epsilon(\mathbf{x}, t) \approx w_i'(\mathbf{x}, t), \tag{101}$$

$$u_i'(\mathbf{x}, t) := \varphi_{ij}(\mathbf{x})\bar{u}_j(\mathbf{m}, t) + \psi_{ij}(\mathbf{x})\dot{\bar{w}}_j(\mathbf{m}, t), \mathbf{x} \in F_{\mathbf{m}}, \tag{102}$$

$$w_i'(\mathbf{x}, t) := \bar{w}_i(\mathbf{m}, t), \mathbf{x} \in S_{\mathbf{m}}. \tag{103}$$

Here, φ_{ij} and ψ_{ij} are the local trial functions within the cell with index $\mathbf{m}$. They describe the local fluid velocity in the direction of x_i due to average fluid or rod velocities in the direction of x_j within any cell. As a consequence of the variational principle they satisfy

$$\varphi_{ij} = \partial \Phi_j / \partial x_i, \qquad \partial^2 \Phi_j / \partial x_i^2 = 0, \tag{104}$$

$$\psi_{ij} = \partial \Psi_j / \partial x_i, \qquad \partial^2 \Psi_j / \partial x_i^2 = 0, \tag{105}$$

which is consistent with the assumption of *locally* incompressible and inviscid flow.

The solutions are normalized by

$$^i\bar{\varphi}_{ij} = \delta_{ij}, \qquad ^i\bar{\psi}_{ij} = 0. \tag{106}$$

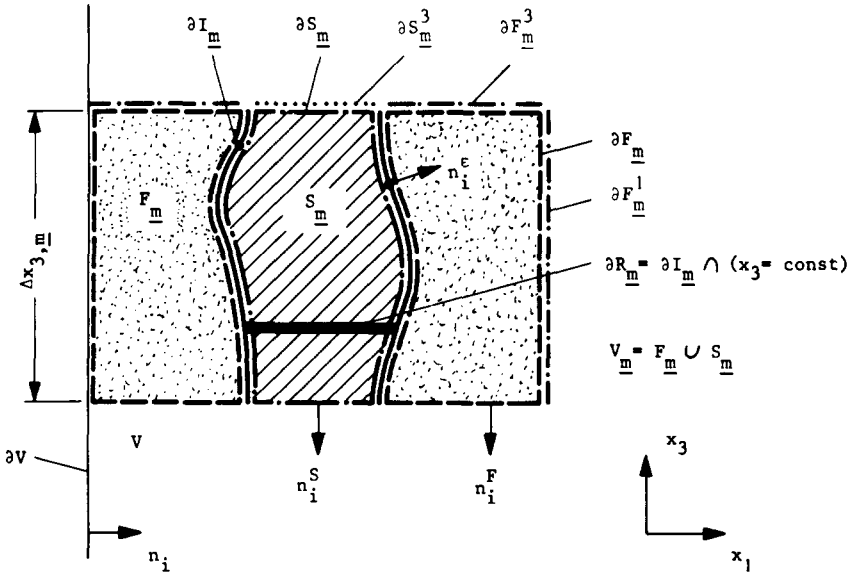

Fig. 2. Domain notations.

Here

$$i\bar{y} := \iint_{\partial F_m{}^i} y \, d0 / |\partial F_m{}^i| \qquad [107]$$

is the surface mean value of any quantity y in the fluid, averaged over the fluid part of the cell with unit outward normal in the direction of the coordinate x_i. Due to this normalization

$$\overset{i-1}{u_i} = \bar{u}_i, \qquad [108]$$

so that $\bar{u}_i$ represents the surface mean velocity while $\bar{w}_i$ is the volume mean displacement which equals the corresponding surface mean displacement. At the fluid-structure interface ∂I_m with unit normal n_i^ϵ the normal velocity must be continuous. As a consequence of [102, 103] the trial functions have to satisfy

$$n_i^\epsilon \varphi_{ij} = 0, \qquad n_i^\epsilon \psi_{ij} = n_j^\epsilon \qquad [109]$$

at ∂I_m. Further, φ_{ij} and ψ_{ij} have to be periodic functions of x_i with periods ϵ^i, $i = 1, 2, 3$. This condition is necessary to insure that u_i' becomes a continuous space function in the fluid domain for $\epsilon/D \to 0$. These statements have been explained in Schumann (1981).

Now, consider any plane $\partial F'^i$ within a cell and parallel to ∂F^i as sketched in figure 3. The magnitude $|\partial F'^i|$ differs in general from $|\partial F^i|$. As φ_{ij} describes a solenoidal flow field for fixed rods, the flow described by φ_{ij} across $\partial F'^i$ and ∂F^i must be equal. Therefore,

$$\iint_{\partial F_m'{}^i} \varphi_{ij} \, d0 = \iint_{\partial F_m{}^i} \varphi_{ij} \, d0 = {}^i\bar{\varphi}_{ij} |\partial F_m{}^i| \qquad [110]$$

for all planes $\partial F_m'{}^i$ but in general

$$\iint_{\partial F_m'{}^i} \varphi_{ij} \, d0 / |\partial F_m'{}^i| \neq \iint_{\partial F_m{}^i} \varphi_{ij} \, d0 / |\partial F_m{}^i|$$

Thus the normalization (not the form) of φ_{ij} and consequently of $\bar{u}_i$ changes if we use a shifted cell definition. This means that these quantities are dependent on the cell definition.

It is easy to show that

$$\psi_{ij} = \delta_{ij} - \varphi_{ij}, \qquad [111]$$

because the r.h.s. satisfies all conditions [105, 106, 109] on ψ_{ij}. Thus ψ_{ij} is cell-dependent too. But, while φ_{ij} is cell-dependent only with respect to its magnitude, ψ_{ij} is cell-dependent also in its form. The reason for this cell-dependence can be understood from figure 4. The surface mean flow related with ψ_{ij} across the faces ∂F^i is zero while the rod moves. However, the volume mean values of ψ_{ij} within the cell is generally non-zero because of the fluid displaced by the rod.

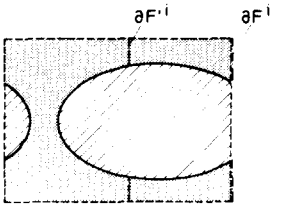

Fig. 3. Definition of a surface $\partial F'^i$ in the fluid which is parallel to the surface ∂F^i.

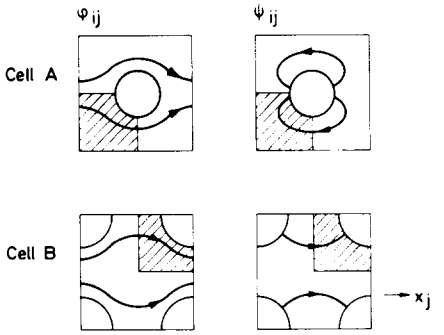

Fig. 4. Sketches of the flow fields φ_{ij} and ψ_{ij} for different cell types. The shaded domains have equal form. One can see, that the flow pattern of φ_{ij}, but not that of ψ_{ij}, is equal for both cell types.

Actually, the volume mean fluid velocity is

$$\bar{u}_i^* := \iiint_{F_m} u_i' \, \mathrm{d}V/|F_m|, \qquad [112]$$

$$= \iiint_{F_m} (\varphi_{ij}\bar{u}_j + (\delta_{ij} - \varphi_{ij})\dot{\bar{w}}_j) \, \mathrm{d}V/|F_m| .$$

Because of [110]

$$\iiint_{F_m} \varphi_{ij} \, \mathrm{d}V = \int_{\epsilon^i} \iint_{\partial F'^i} \varphi_{ij} \, \mathrm{d}0 \, \mathrm{d}x_i$$

$$= \epsilon^i \bar{\varphi}_{ij} |\partial F_m{}^i| = \epsilon^i |\partial F_m{}^i| \delta_{ij} ,$$

so that

$$\iiint_{F_m} \varphi_{ij} \, \mathrm{d}V/|F_m| = \epsilon^i |\partial F_m{}^i| \delta_{ij} / (\alpha \epsilon^i |\partial V_m{}^i|)$$

$$= \delta_{ij} \gamma^i / \alpha . \qquad [113]$$

Here $\alpha := |F_m|/|V_m|$ is the fluid volume fraction and $\gamma^i := |\partial F_m{}^i|/|\partial V_m{}^i|$ is the permeability. Thus

$$\bar{u}_i^* = [\gamma^i \bar{u}_i + (\alpha - \gamma^i)\dot{\bar{w}}_i]/\alpha . \qquad [114]$$

This shows that volume and surface average velocities are equal if and only if $\alpha = \gamma^i$, $i = 1, 2, 3$. This is the case if the cell type is defined accordingly or if we have a random phase distribution.

For notation consistency, we set

$$\bar{w}_i^* \equiv \bar{w}_i \qquad [115]$$

and replace $\bar{u}_i$ and $\bar{w}_i$ by the new variables $\bar{u}_i^*$, $\bar{w}_i^*$ using [114, 115]. We then have the alternative ansatz

$$u_i' = \varphi_{ij}^* \bar{u}_j^* + \psi_{ij}^* \bar{w}_j^* \qquad [116]$$

with

$$\varphi_{ij}^{*} := (\alpha/\gamma^{i})\varphi_{ij} \qquad [117]$$

$$\psi_{ij}^{*} := \delta_{ij} - \varphi_{ij}^{*} . \qquad [118]$$

In the limit $\epsilon/D \to 0$ the discrete fields $\bar{u}_j^*(\mathbf{m}, t)$, $\bar{w}_j^*(\mathbf{m}, t)$ are replaced by smooth functions $u_j^*(\mathbf{x}, t)$, $w_j^*(\mathbf{x}, t)$.

3. AVERAGED CONTINUITY EQUATION

The averaged continuity equations express the averaged rate of change of density ρ and ρ_s in the cell due to the inflowing material. This flow is intrinsically related to surface averages. It was for this reason that surface averages have been used in Schumann (1981). The result was

$$\frac{\partial}{\partial t}(\alpha\rho) = -\rho_0 \frac{\partial}{\partial x_i}(\gamma^i u_i) \qquad [119]$$

$$\frac{\partial}{\partial t}[(1-\alpha)\rho_s] = -\rho_s{}^0 \frac{\partial}{\partial x_i}[(1-\gamma^i)\dot{w}_i^*] . \qquad [120]$$

Here, ρ_0 and $\rho_s{}^0$ are the nomial microscopic densities of the fluid and structure material, respectively.

If [114] is used to replace u_i one obtains from [119]

$$\frac{\partial}{\partial t}(\alpha\rho) = -\rho_0 \frac{\partial}{\partial x_i}[\alpha u_i^* - (\alpha - \gamma^i)\dot{w}_i^*] . \qquad [121]$$

From [120, 121] one can eliminate $\partial\alpha/\partial t$ and obtains

$$\frac{\alpha}{\rho_0}\frac{\partial\rho}{\partial t} + \frac{1-\alpha}{\rho_s^0}\frac{\partial\rho_s}{\partial t} + \frac{\partial}{\partial x_i}[\alpha u_i^* + (1-\alpha)\dot{w}_i^*] = 0 . \qquad [122]$$

We note that the continuity eqns [120, 121] for the two individual phases still depend on the surface permeabilities and are, therefore, cell-dependent. One has, therefore, to be very careful in the definition of this cell. This problem has been also discussed by Reed & Stewart (1980). The common form (Ishii 1975) of two-phase continuity equations

$$\frac{\partial}{\partial t}(\alpha\rho) + \frac{\partial}{\partial x_i}(\alpha\rho u_i^*) = 0 \qquad [121a]$$

$$\frac{\partial}{\partial t}[(1-\alpha)\rho_s] + \frac{\partial}{\partial x_i}[(1-\alpha)\rho_s\dot{w}_i^*] = 0 \qquad [120a]$$

is obtained if we replace the local derivative $\partial/\partial t$ by the substantial derivative and if we restrict ourselves to cases with $\alpha = \gamma^i$, $i = 1, 2, 3$.

4. HOMOGENIZED EQUATIONS IN TERMS OF VOLUME AVERAGED VELOCITIES

One could deduce the homogenized equations of motion in terms of the volume averaged velocities by introducing [114, 115] in the analogous equations [41] for surface averaged velocities. However, the result would include asymmetric tensors unless the suitable linear combination of the single equation components is taken. It is easier, therefore, to start again

from Hamilton's principle, introducing the new ansatz [116]. With $\dot{v}_i^* \equiv u_i^*$ the variation of the functional after partial integration in time gives

$$\delta\bar{J} = \int_{t_1}^{t_2} \sum_{\mathbf{m}} \left\{ \iiint_{F_{\mathbf{m}}} [-\rho_0(\varphi_{ij}^* \ddot{\bar{v}}_j^* + \psi_{ij}^* \ddot{\bar{w}}_j^*) + \rho_0 g_i^\epsilon] \times [\varphi_{ik}^* \delta\bar{v}_k^* + \psi_{ik}^* \delta\bar{w}_k^*] \, \mathrm{d}V + \int_{\Delta x_{3,\mathbf{m}}} [-M_{ij}^\epsilon \ddot{\bar{w}}_j^* + f_i^\epsilon] \, \mathrm{d}x_3 \delta\bar{w}_i^* \right\} \mathrm{d}t. \quad [123]$$

Here, $\Delta x_{3,\mathbf{m}}$ is the interval taken by cell $\mathbf{m}$ on the axis x_3, M_{ij}^ϵ is a diagonal tensor of rod inertia per unit x_3-coordinate length, g_i^ϵ is the local force per unit fluid mass, and f_i^ϵ is the local force on the structure per unit x_3-coordinate length.

As φ_{ij}^*, ψ_{ij}^* are known functions, we introduce abbreviations which have the meaning of effective densities and forces per unit volume:

$$\rho_{ij}^{*ff} := \frac{1}{|V_{\mathbf{m}}|} \iiint_{F_{\mathbf{m}}} \rho_0 \varphi_{ki}^* \varphi_{kj}^* \, \mathrm{d}V \quad [124a]$$

$$\rho_{ij}^{*fs} \equiv \rho_{ji}^{*sf} := \frac{1}{|V_{\mathbf{m}}|} \iiint_{F_{\mathbf{m}}} \rho_0 \varphi_{ki}^* \psi_{kj}^* \, \mathrm{d}V \quad [124b]$$

$$\rho_{ij}^{*ss} := \frac{1}{|V_{\mathbf{m}}|} \iiint_{F_{\mathbf{m}}} \rho_0 \psi_{ki}^* \psi_{kj}^* \, \mathrm{d}V \quad [124c]$$

$$m_{ij} := \frac{1}{|V_{\mathbf{m}}|} \int_{\Delta x_{3,\mathbf{m}}} M_{ij}^\epsilon \, \mathrm{d}x_3 \equiv (1-\alpha)\rho_s^0 \delta_{ij} \quad [124d]$$

$$g_i^* := \lim_{\epsilon/D \to 0} \frac{1}{|V_{\mathbf{m}}|} \iiint_{F_{\mathbf{m}}} \rho_0 g_j^\epsilon \varphi_{ji} \, \mathrm{d}V \quad [125a]$$

$$f_i^* := \lim_{\epsilon/D \to 0} \frac{1}{|V_{\mathbf{m}}|} \left[\iiint_{F_{\mathbf{m}}} \rho_0 g_i^\epsilon \, \mathrm{d}V + \int_{\Delta x_{3,\mathbf{m}}} f_i^\epsilon \, \mathrm{d}x_3 \right]. \quad [125b]$$

Because of $\psi_{ij}^* = \delta_{ij} - \varphi_{ij}^*$ the density tensors are related to one virtual density tensor

$$\rho_{ij}^V := \frac{1}{|V_{\mathbf{m}}|} \iiint_{F_{\mathbf{m}}} \rho_0 \varphi_{ki}^* (\varphi_{kj}^* - \delta_{kj}) \, \mathrm{d}V \quad [126]$$

by

$$\rho_{ij}^{*ff} = \alpha\rho_0\delta_{ij} + \rho_{ij}^V \quad [127a]$$

$$\rho_{ij}^{*fs} = \rho_{ij}^{*sf} = -\rho_{ij}^V \quad [127b]$$

$$\rho_{ij}^{*ss} = \rho_{ij}^V. \quad [127c]$$

It is easy to see from [124a] that ρ_{ij}^{*ff} is a symmetric tensor. From [127] follows that all other tensors are symmetric as well.

The homogenized equations of motion are obtained as in Schumann (1981) from the demand

$\delta\bar{J} = 0$. Hereby, the constraint

$$\partial[\alpha(\delta\dot{v}_i^*) + (1-\alpha)(\delta\dot{w}_i^*)]/\partial x_i = 0\,,$$

which is a consequence of [122] for fixed density, is taken into account by the averaged pressure p as Lagrangian multiplier. The result is

$$\alpha\rho_0\dot{u}_i^* + \rho_{ij}^V(\dot{u}_j^* - \ddot{w}_j^*) = g_i^* - \alpha\partial p/\partial x_i, \qquad [129a]$$

$$(1-\alpha)\rho_s^0\ddot{w}_i^* + \rho_{ij}^V(\ddot{w}_j^* - \dot{u}_j^*) = -g_i^* + f_i^* - (1-\alpha)\partial p/\partial x_i\,. \qquad [129b]$$

The boundary conditions prescribe $p = p_0$ on one part $\partial V_p \subset \partial V$ of the outer boundary ∂V and $\alpha n_i u_i^* + (1-\alpha)n_i\dot{w}_i^* = u_n{}^b$ on the rest $\partial V - \partial V_p$, where

$$u_n{}^b := \alpha u_n^* + (1-\alpha)\dot{w}_n^* \qquad [130a]$$

$$= \gamma^n u_n + (1-\gamma^n)\dot{w}_n. \qquad [130b]$$

The index n refers to the normal direction on ∂V and u_n, $\dot{w}_n$ are surface averaged velocities as defined in [40] whereas u_n^*, $\dot{w}_n^*$ are volume averaged normal velocities in the cells adjacent to the outer boundary ∂V.

As φ_{ij}^* and ψ_{ij}^* are independent of the cell type, all terms in [129] are also independent of the cell type chosen. In the sum of eqns [129a] and [129b] the relative acceleration drops out. This is consistent with Newton's third principle of equality of action and reaction. In fact, the form of [129] is quite conventional (Drew *et al.* 1979). This was not the case in the previous publication (Schumann 1981). The essential points are the manner how this equation was deduced, the fact that the virtual density can be evaluated numerically, and that it is cell type independent.

5. SOLUTIONS AND SOUND SPEEDS

Equations [129] can be solved explicitly for the accelerations $\dot{u}_i^*$ and $\ddot{w}_i^*$. For brevity we assume that the virtual density tensor is a diagonal tensor. This condition can be achieved by a coordinate transformation. Thus, we can set

$$\rho_{ij}^V = \rho_0\bar{\rho}^i\delta_{ij}, \qquad [131]$$

where $\bar{\rho}^i$ is a non-dimensional virtual density. Further, let $\kappa := \rho_s^0/\rho_0$. Then the solutions are

$$\dot{u}_i^* = \{-[\bar{\rho}^i + \alpha(1-\alpha)\kappa]\partial p/\partial x_i + (1-\alpha)\kappa g_i^* + \bar{\rho}^i f_i^*\}/\det \qquad [132a]$$

$$\ddot{w}_i^* = \{-[\alpha(1-\alpha) + \bar{\rho}^i]\partial p/\partial x_i - \alpha g_i^* + (\alpha + \bar{\rho}^i)f_i^*\}/\det \qquad [132b]$$

with

$$\det := \rho_0[(\alpha + \bar{\rho}^i)(1-\alpha)\kappa + \alpha\bar{\rho}^i]. \qquad [132c]$$

For zero external forces ($g_i^* = f_i^* \equiv 0$) one can deduce the acceleration ratio

$$\frac{\ddot{w}_i^*}{\dot{u}_i^*} = \frac{\alpha(1-\alpha) + \bar{\rho}^i}{\kappa\alpha(1-\alpha) + \bar{\rho}^i} = \begin{cases} 1 & \text{for } \bar{\rho}^i \to \infty \\ 1/\kappa & \text{for } \bar{\rho}^i \to 0 \end{cases}, \qquad [133]$$

and the effective density ρ^i_{eff} which is defined by

$$\rho^i_{\text{eff}}(\alpha\ddot{u}^*_i + (1-\alpha)\ddot{w}^*_i) \equiv \alpha\rho_0\ddot{u}^*_i + (1-\alpha)\rho_i^0\ddot{w}^*_i\,, \qquad [134]$$

with the result

$$\rho^i_{\text{eff}} = \frac{(\alpha+\bar{\rho}^i)(1-\alpha)\kappa + \alpha\bar{\rho}^i}{\alpha(1-\alpha)[\alpha\kappa+(1-\alpha)]+\bar{\rho}^i}\,\rho_0 \qquad [135]$$

$$= \begin{cases} \alpha\rho_0 + (1-\alpha)\rho_s^0 & \text{for } \bar{\rho}^i \to \infty \qquad [136a] \\ \left[\dfrac{\alpha}{\rho_0} + \dfrac{(1-\alpha)}{\rho_s^0}\right]^{-1} & \text{for } \bar{\rho}^i \to 0\,. \qquad [136b] \end{cases}$$

The limiting values correspond to layered two-phase mixtures, where the layers are in series ($\bar{\rho}^i \to \infty$) or in parallel ($\bar{\rho}^i \to 0$), see figure 5. As illustrated in this figure, the case $\bar{\rho} \to \infty$ means that no relative motion between the two phases is possible within the cell. Because of the compressibility of the mixture this does not exclude relative motions on large scales as discussed, e.g. by Sun *et al.* (1968) and Hegemier & Nayfeh (1973).

The results [132] can be used to derive the pressure wave equation from the time derivative of the continuity equation [122] and the averaged equations of state [32]:

$$\ddot{p} - \frac{\partial}{\partial x_i}(A_{ij}\partial p/\partial x_j) = \partial r_i/\partial x_i\,, \qquad [137a]$$

$$r_i = -\{\alpha(1-\alpha)(\kappa-1)g^*_i + [\bar{\rho}^i + \alpha(1-\alpha)]f^*_i\} \qquad [137b]$$

$$/\left\{\det\left[\frac{\alpha}{\rho_0 a^2} + \frac{1-\alpha}{\rho_s^0 a_s^2}\right]\right\}.$$

Here, a and a_s are the speed of sound due to compression in the fluid and structure, respectively, and

$$A_{ij} \equiv (a^i_{\text{eff}})^2\delta_{ij} \qquad [137c]$$

is a tensor containing the squared sound speeds which have the values

$$a^i_{\text{eff}} = \left\{\rho^i_{\text{eff}}\left[\frac{\alpha}{\rho_0 a^2} + \frac{1-\alpha}{\rho_s^0 a_s^2}\right]\right\}^{-1/2}. \qquad [138]$$

Limiting values of this expression are

$$a^i_{\text{eff}} = \begin{cases} \left\{[\alpha\rho_0 + (1-\alpha)\rho_s^0]\left[\dfrac{\alpha}{\rho_s^0 a^2} + \dfrac{1-\alpha}{\rho_s^0 a_s^2}\right]\right\}^{-1/2} & [139a] \\ \text{for } \bar{\rho}^i \to \infty, & \\ \left[\dfrac{\alpha}{\rho_0} + \dfrac{1-\alpha}{\rho_s^0}\right]^{1/2}\left[\dfrac{\alpha}{\rho_0 a^2} + \dfrac{1-\alpha}{\rho_s^0 a_s^2}\right]^{-1/2} & [139b] \\ \text{for } \bar{\rho}^i \to 0\,, & \end{cases}$$

and

$$a^i_{\text{eff}} = \begin{cases} [a^2\alpha/(\alpha+\bar{\rho}^i)]^{1/2} & [140a] \\ \text{for} \quad \kappa \to \infty\,, & \\ \{\kappa a_s^2[\alpha(1-\alpha)+\bar{\rho}^i]/[\alpha(1-\alpha)]\}^{1/2} & [140b] \\ \text{for} \quad \kappa \to 0\,. & \end{cases}$$

The limit $\kappa \to \infty$ corresponds to very heavy or even fixed rods, the limit $\kappa \to 0$ is that of gas bubbles (of fixed form) in a fluid.

For a bundle of circular cylindrical rods in a square pattern with axis in x_3 direction we can reformulate the result of Schumann (1981), see appendix, as

$$\rho^V_{ij} = \text{diag}\,(\bar{\rho}^1, \bar{\rho}^1, 0) \qquad [141]$$

with

$$\bar{\rho}^1 = k(\alpha) \cdot \alpha(1-\alpha) \qquad [142]$$

$$k(\alpha) \approx 1 \quad \text{for} \quad (1-\alpha) \ll 1\,.$$

The values of $\bar{\rho}^1(\alpha)$ and $k(\alpha)$ are plotted in figure 6. For $\alpha = 1 - \pi/4$ the virtual density in the direction normal to the rod axes becomes infinite because then the rods touch each other and close the flow path.

6. COMPARISON WITH THE LITERATURE

The effect of virtual masses in rod bundles has been treated in the past, e.g. by Chen (1978), by setting up the virtual fluid mass matrix which couples the acceleration of a rod with those of neighbouring rods. This matrix becomes very large for extensive rod bundles and cannot be

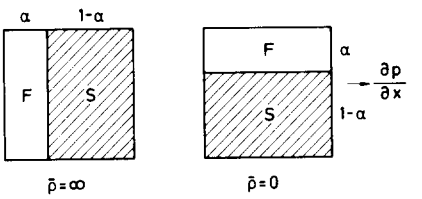

Fig. 5. Extreme cases with infinite and zero virtual density $\bar{\rho}$.

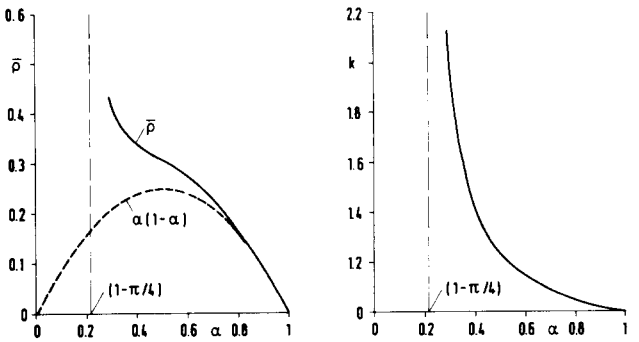

Fig. 6. Virtual density $\bar{\rho} = k(\alpha)\,\alpha(1-\alpha)$ for a cylinder in a square pattern as a function of the fluid volume fraction α and the correction factor $k(\alpha)$.

computed explicitely, therefore, for large bundles. The concept of virtual densities as introduced by Schumann (1981) has not been discussed for rod bundles, so far. However, related approaches have been used for porous bodies and for spherical particles floating homogeneously distributed in a fluid. It is interesting to relate our theory to such investigations.

A general homogenization theory for porous bodies has been deduced by Biot (1956, 1962). His approach is similar to the present one in that he too starts from Hamilton's principle. He identifies the densities and in particular the virtual density which describes the inertia coupling between the phases. However, because of the assumed statistical homogeneity of the porous body he does not have to distinguish between surface and volume average. Also, the theoretical expressions cannot be evaluated numerically because of the unspecified body geometry. Further, the present approach has many elements in common with the micro-structure theories of Mindlin (1964), Sun *et al.* (1968), and Drumheller & Bedford (1980) and others, in particular in the sense that these theories, too, use Hamilton's principle. Some of these theories are far more refined than the present one in that not only translational relative motions are considered but also other modes of motion including such with large convective accelerations. On the other hand they are either limited to one-dimensional laminated composites or have not been evaluated quantitatively. Recently, Fleury (1980) derived a set of homogenized equations following the mathematics of Bensoussan *et al.* (1977, 1978). His result is formulated in terms of a single average velocity vector and an inverse density tensor. He does not give explicit expressions for specific geometries and one cannot directly compare his result with Biot's and the present ones. Some empirical relations for the virtual density in porous bodies are discussed by Barzam (1980).

For spherical particles in fluid several authors, e.g. Batchelor (1969), Crespo (1969), Drew *et al.* (1979), use the simple formula

$$\bar{\rho}^i = \frac{1}{2}\bar{\alpha},\ \bar{\alpha} := 1-\alpha,\quad i=1,2,3 \qquad [143]$$

which is exact for potential flow and void fractions $\bar{\alpha} \ll 1$. Morris & Stewart (1976) discuss the differences between spherical gas bubbles and solid particles. They state that for high frequencies (Strouhal numbers) both have the same virtual mass.

For large values of void fractions $\bar{\alpha}$, only approximate results are known. Obviously it is not simple to solve Laplace's equation for a system of spheres. Zuber (1964) and Oshima (1979) computed the virtual density by using the analytical potential flow solution for a single sphere accelerated within a concentric outer sphere. The result of Zuber has also been deduced by Wijngaarden (1976),

$$\bar{\rho}^i = \frac{1}{2}\bar{\alpha}(1+2\bar{\alpha})/(1-\bar{\alpha}),\quad i=1,2,3 \qquad [144]$$

and is used, e.g. by Mecredy & Hamilton (1972) and Ardron & Duffey (1978). Oshima's result is

$$\bar{\rho}^i = \frac{1}{2}\bar{\alpha}(1-\bar{\alpha}),\quad i=1,2,3\,. \qquad [145]$$

The present author cannot explain the reasons for the differences. Oshima's result is attractive because it is very similar to the result [142] for cylinders. For a random suspension of spherical particles Wijngaarden (1976) has found numerically the approximate relation

$$\bar{\rho}^i = \frac{1}{2}\bar{\alpha}(1+2.78\bar{\alpha}) + 0(\bar{\alpha}^2)\,. \qquad [146]$$

Srebnyuk & Gorban (1979) have computed the virtual mass for an individual accelerated sphere within a finite set of neighbouring spheres lying on a line or in a plane. Because of the restricted number of spheres and the non-smooth motion type these results cannot be related to the present theory.

Thus, in summary, the exact dependence of the virtual density in a suspension of spherical particles for large void ratios $\bar{\alpha}$ is still unknown. Of particular interest would be the limiting value for a system of closely packed spheres which, for a square arrangement, appears if $\bar{\alpha} = \pi/6$. In contrast to the cylindrical case, the virtual density is still finite for this limiting state.

The experiment of Carlucci (1975) can be taken to support the result that the effective density is reduced in a two-phase mixture if relative acceleration is possible between the phases. Carlucci measured the effective hydrodynamic mass m_H of a rod oscillating normal to its axis. The rod is located concentrically inside of a rigid flow tube. The annular region between the inner and outer cylinder wall is filled by a flowing air–water two-phase mixture. For a single phase fluid the value of the effective mass is well known and proportional to the fluid density. For a homogeneous two-phase mixture without slip the effective mass should decrease linearly with the void fraction $\bar{\alpha}$. However, Carlucci's results, see figure 7, show that the effective mass is smaller than expected. Carlucci suggests that the deviation is due to the non-uniform density distribution across the annulus or due to compressibility effects. An alternative explanation is as follows. Because of the relative acceleration between the phases the effective density is ρ_{eff} as given by [135] which for finite virtual density is less than the volumetrically averaged mixture density. This can be seen from figure 7, where the effective mass has been plotted as computed from [135] and the approximate virtual density formulas [145, 144], i.e.

$$\frac{\bar{\rho}_{\text{eff}}}{\rho_0} = \begin{cases} \dfrac{1-\bar{\alpha}}{2\bar{\alpha}+1}, & \text{from [145]}, \quad [147a] \\ \dfrac{(1-\bar{\alpha})(1+2\bar{\alpha})}{1+4\bar{\alpha}-2\bar{\alpha}^2}, & \text{from [144]}, \quad [147b] \end{cases}$$

taking $\kappa = 0$. Of course, the underlying assumption of small spherical air bubbles in potential flow is an over-simplification in particular for large void fractions. Further, at very small void fractions viscous forces reduce the slip between the phases so that it is clear that the theoretical

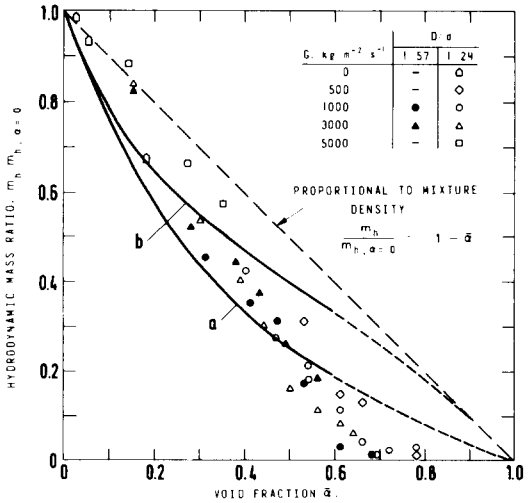

Fig. 7. Variation of hydrodynamic effective mass with void fraction $\bar{\alpha} = 1 - \alpha$. Experimental points of Carlucci (1979) and theoretical curves corresponding [147a/b].

curves underestimate the effective density in this range. Otherwise we see that [147] envelopes the experimental data, which supports the given explanation.

Several authors give expressions for the effective sound speed in a two-phase mixture like air–water mixtures. It appears as if the close relation between the sound speed and the effective density and thus the virtual density as given in [138] has not been always noted. Limiting values like the "equilibrium" sound speed for $\bar{\rho}^i = \infty$ or the "stratified" or "frozen" sound speed for $\bar{\rho}^i = 0$, eqn [139], agree with those given by Chawla & Böckh (1971), Mecredy & Hamilton (1972), and Ardron & Duffey (1978).

Böckh & Chawla (1974), and Böckh (1975) report experimental results on the speed of sound. They find that the bulk of large amplitude pressure waves in air-water mixtures travels with the equilibrium sound speed [139a] while the small amplitude foot of the pressure waves shows a larger sound speed. This difference can be attributed to friction effects which reduce the slip between the phases in the bulk of the wave.

Parker (1978) gives without derivation the formula $a_{\text{eff}} = a/(2-\alpha)^{1/2}$. He verifies experimentally this relation for fixed solid rods ($\kappa \to \infty$). The formula is consistent with the present result [140a, 142] for $(1-\alpha) \ll 1$.

The acceleration ratio [133] is consistent with the result of Batchelor (1969) and Morrison & Stewart (1976) who deduced a maximum ratio of three for $\kappa = 0$ and large frequency motion. The value three is peculiar to small spherical particles. For cylinders accelerated normal to their axis this limiting value is two, as can be seen from [133, 142]. The acceleration ratio [133] for $\bar{\rho} \to 0$ is also the maximum possible velocity ratio. It is interesting to note that for choking two-phase flow the condition of maximum momentum or energy transport led Fauske (1961) and Moody (1969) and others to the postulate that the velocity ratio should be given as 1/2 or 1/3 power of this maximum acceleration ratio.

In conclusion, our results are consistent with the literature. The main achievement is a theory which predicts effective densities, acceleration ratios and speeds of sound for the whole range of physically possible virtual density values for two-phase mixtures with a random or periodic phase distribution.

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APPENDIX: RELATION TO HOMOGENIZED EQUATIONS IN TERMS OF SURFACE AVERAGES

The density tensors and forces defined in [34] of Schumann (1981) are related to the quantities defined in [124, 125] due to [117] with $\mu^{ij} := \gamma^i\gamma^j/\alpha^2$ as

$$\begin{aligned}
\rho_{ij}^{ff} &= \mu^{ij}(\rho_{ij}^{V} + \alpha\delta_{ij}\rho_0),\\
\rho_{ij}^{fs} &= \rho_{ij}^{sf} = -\mu^{ij}\rho_{ij}^{V} + \rho_0\delta_{ij}(\gamma^i - \alpha\mu^{ij}),\\
\rho_{ij}^{ss} &= m_{ij} + \mu^{ij}\rho_{ij}^{V} + \rho_0\delta_{ij}(\alpha - \gamma^j - \gamma^i + \mu^{ij}\alpha).\\
g_i &= (\gamma^i/\alpha)g_i^*,\\
f_i &= f_i^* - g_i.
\end{aligned} \tag{A1}$$

We note that for $\alpha = \gamma^i$ all factors α, γ^i cancel to unit factors or zero summands.

For case of circular cylindrical rods in a square pattern the result of the previous paper [47] can be restated as

$$\rho_{ij}^{V} = \operatorname{diag}(\rho_0\bar{\rho}^1, \rho_0\bar{\rho}^2, 0) \tag{A2}$$

with

$$\begin{aligned}
\bar{\rho}^1 = \bar{\rho}^2 &= \alpha[(2-\alpha)k_{ff} - 1]\\
&\approx \alpha(1-\alpha) \quad \text{for} \quad (1-\alpha) \ll 1.
\end{aligned} \tag{A3}$$

Here k_{ff} is the correction factor plotted in figure 5 of Schumann (1981). In fact, because all four tensors are related to ρ_{ij}^{V}, the different correction factors shown in this figure are not independent but satisfy

$$\begin{aligned}
(2-\alpha)(k_{ff}-1) + 1 - \alpha &= (1-\alpha)[2(k_{fs}-1)+1]\\
&= (1-\alpha)[(2-\alpha)(k_{ss}-1)+1],\\
k_{ff} &\le k_{fs} \le k_{ss}.
\end{aligned} \tag{A4}$$